Math 125B
Vogler

Riemann Integration on Jordan Regions

Defn Let $E \subseteq \mathbb{R}^n$ be a Jordan region, let $f: E \to \mathbb{R}$ be bounded, $R \supseteq E$ be n-dimensional rectangle, and $\mathcal{G} := \{R_1, \ldots, R_p\} \in \mathcal{P}(R)$. Extend f to $\mathbb{R}^n$ by setting $f(\vec{x}) = 0 \;\; \forall \vec{x} \in \mathbb{R}^n - E$.

1) The <u>upper sum</u> of f on E with respect to $\mathcal{G}$ is
$$U(f,\mathcal{G}) := \sum_{R_j \cap E \neq \emptyset} M_j |R_j|,$$
where $M_j := \sup_{\vec{x} \in R_j} f(\vec{x}) \;\; \forall j = 1, \ldots, p$.

2) The <u>lower sum</u> of f on E with respect to $\mathcal{G}$ is
$$L(f,\mathcal{G}) := \sum_{R_j \cap E \neq \emptyset} m_j |R_j|,$$
where $m_j := \inf_{\vec{x} \in R_j} f(\vec{x}) \;\; \forall j = 1, \ldots, p$.

3) The <u>upper integral</u> of f on E is
$$\overline{\int_E} f(\vec{x})\, d\vec{x} := \inf_{\mathcal{G} \in \mathcal{P}(R)} U(f,\mathcal{G}).$$

4) The <u>lower integral</u> of f on E is
$$\underline{\int_E} f(\vec{x})\, d\vec{x} := \sup_{\mathcal{G} \in \mathcal{P}(R)} L(f,\mathcal{G}).$$

-Notes: a) These are the natural extensions of integration of 1 variable, compare with definitions 5.3 and 5.13.

b) If $\mathcal{G}, \mathcal{H} \in \mathcal{P}(R)$, then $L(f,\mathcal{G}) \leq U(f,\mathcal{H})$.

c) Like the 1 variable versions, the upper and lower integrals always exist and satisfies $\underline{\int_E} f(\vec{x})\,d\vec{x} \leq \overline{\int_E} f(\vec{x})\,d\vec{x}$

<u>Notation</u> Unless otherwise stated, $E \subseteq \mathbb{R}^n$ is a Jordan region, $R \supseteq E$ is a bounding n-dimensional rectangle, $\mathcal{G} := \{R_1, \ldots, R_p\} \in \mathcal{P}(R)$ is a grid with p rectangles.

Defn A function $f: E \to \mathbb{R}$ is Riemann integrable on E, denoted $f \in \mathcal{R}(E)$, if f is bounded and $\forall \varepsilon > 0, \exists G \in \mathcal{P}(R)$ such that
$$U(f, G) - L(f, G) < \varepsilon.$$

-Notes: a) Compare this to definition 5.9.

b) One can show that f is integrable if and only if
$$I := \overline{\int_E} f(\vec{x})\, d\vec{x} = \underline{\int_E} f(\vec{x})\, d\vec{x},$$ which is extension of Thm 5.15.

c) If $f \in \mathcal{R}(E)$, we denote the integral by
$$\int_E f(\vec{x})\, d\vec{x} \text{ or } \int_E f\, dV,$$ which is equal to I in b).

Thm (12.20) Let $f: E \to \mathbb{R}$ be bounded. Then, $\forall \varepsilon > 0, \exists G_0 \in \mathcal{P}(R)$ such that when $G := \{R_1, \dots, R_p\} \in \mathcal{P}(I)$ is finer than G_0, we have
$$\left| \overline{\int_E} f(\vec{x})\, d\vec{x} - \sum_{R_j \subset E^o} M_j |R_j| \right| < \varepsilon$$
and
$$\left| \underline{\int_E} f(\vec{x})\, d\vec{x} - \sum_{R_j \subset E^o} m_j |R_j| \right| < \varepsilon,$$
where $M_j := \sup_{\vec{x} \in R_j} f(\vec{x})$ and $m_j = \inf_{\vec{x} \in R_j} f(\vec{x})$ $\forall j = 1, \dots, p$.

Note: This is telling us that boundry values do not contribute any thing to the integral, which makes sense because they lack volume since E is a Jordan region.

Thm (12.21)
If $E \subseteq \mathbb{R}^n$ is a closed Jordan region and $f: E \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}(E)$.

Thm (12.22) $\quad Vol(E) = \int_E 1\, d\vec{x}.$